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Poisson coefficient of open cellular solids subject to structural anisotropy

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Abstract The Poisson coefficients of a 3D cellular solid are calculated as function of structure characteristics. A significant structural anisotropy is introduced by considering ellipsoid voids where their elongation and orientation are controlled using the random sequential addition (RSA) algorithm. The relative density of the material is varied between 0.1 and 0.3 in order to obtain an open cell structure. Finite element calculation is performed to calculate the six Poisson coefficients of the material. Predicted results show that all quantities are not independent and that three independent coefficient sets can be related to the ratio of the fully oriented ellipsoids and the elongation parameters. No clear correlation can be derived between the Poisson coefficients and the relative density suggesting a fixed-point effect. It seems that some coefficients are more sensitive to the Poisson coefficient of the solid phase, especially under those conditions where the percolation of the solid, in the direction of main alignment, is the largest one.

Introduction

In a typical extrusion process, the final microstructure is sensitive to the process variables such as material flow rate, temperature and pressure. In some applications, the process effect is significantly correlated to the void network created during the process. Void alignment in the direction of the material flow or void sizing due to fractioning by screw rotation are some intuitive correlations that can be deduced.

There is a great interest to study the correlations between effective properties of such cellular materials and the void network characteristics. Potential applications are related to the design of shock devices, lightweight materials and thermal barrier coatings, among other examples.

A well-known example of the predicted correlations is the equation relating the elasticity parameters to the relative density of an open cell material [1]. Such equation can be written as:

$$E = C\rho^n,\ \nu = 0.3 \tag{1}$$

where E is the effective Young's modulus (i.e., scaled with respect to solid phase Young's modulus), ρ is the relative density (i.e., the density of the cellular solid over that of the solid phase), C is a constant coefficient that depends on the geometry of the unit cell, n is an exponent and its value depends on the regularity of the network and the cell shape isotropy and ν is the Poisson coefficient, which is thought to be a constant value [1].

Attempts were made to suggest a better description of the Poisson coefficient than that given in Eq. 1. The main motivation is that at large relative densities, the effective Poisson ratio becomes sensitive to the Poisson coefficient of the solid phase. When lowering the relative density, the effect of the geometry of the cells becomes competitive. Thus, a constant value of the effective parameter cannot be guaranteed over a large range of relative densities.

The exact bounds for the Poisson ratio (ν) for porous materials can be derived from the Hashin–Shrikman analytical result related to a composite material containing dilute concentration of inclusions [2]

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$$v = \frac{(3K - 2G)}{(6K + 2G)} \tag{2}$$

where K and G are the effective bulk and shear moduli, respectively.

Thus, the theoretical v range can be deduced by setting $(K > 0, \quad G = 0)$ and $(K = 0, \quad G > 0)$ which gives $-1 \leq v \leq 0.5$. If the material does not exhibit negative Poisson ratio effect, the restricted physical range is $0 \leq v \leq 0.5$.

Zhu et al. [3] Suggest two expressions of the Poisson coefficient as function of the relative density ρ depending on whether the edge cross sections are plateau borders or equilateral triangles

$$v = 0.5 * \left(\frac{1 - \xi\rho}{1 + \xi\rho}\right) \tag{3}$$

where $\xi = 1.514$ is related to plateau border and $\xi = 1.09$ for equilateral triangles.

This result is obtained for a 3D regular set of tetrakaidecahedral cells. Roberts and Garboczi [4] showed that it is possible to obtain a common expression of Poisson ratio for a variety of random cellular solids in the form

$$v = v_s + \left(\frac{1 - \rho}{1 - \rho^0}\right)^k (v_0 - v_s) \tag{4}$$

where ρ is the solid fraction in the cellular solid, (v_0, ρ_0, k) are constants depending on the type of generated cellular solid. Note that in the expression above, the effective Poisson ratio becomes density independent for $v_s = v_0$.

The same authors showed that a percolation limit characterises the effective Poisson ratio (i.e., v becomes independent of v_s) which appears at a fixed point $v_0 = 0.2$.

In the former discussion, the results of cellular structure effect concern an overall estimation of the Poisson ratio. The predictions do not explicit the space anisotropy of the parameter as can be encountered in many situations [5, 6].

In this paper, other geometrical considerations are introduced by using ellipsoid shape voids that represent, in some way, the result of shaping in a typical extrusion process. The aim is to study the six Poisson coefficients, which can be estimated for the cellular solid as function of orientation and alignment of ellipsoids.

Structure generation algorithm

The random sequential addition algorithm or RSA [7, 8] is used to generate 3D cellular solids. The algorithm selects randomly ellipsoid positions and controls their centre-to-centre distance using a criterion called the overlapping distance δ. This criterion is written as:

$$\delta = d_{ij} - (a_i - a_j) \tag{5}$$

The subscribes i and j refer to the ellipsoid labels. d_{ij} is the centre-to-centre distance. a is the minor semi-length of the ellipsoid and is fixed in this study $a = 7.14$.

Equation 5 means that ellipsoids can stay close to each other at the shortest distance δ. Note that this criterion does not depend on space orientation of ellipsoids.

In order to create the open cell structure, δ is set negative ($\delta = -5$ voxels) for the ellipsoids to overlap significantly.

The amount of void in the structure is sensitive to the number of ellipsoids NB and the elongation parameter defined as

$$m = a/b \tag{6}$$

where b is the major semi-axis length (Fig. 1a). m is the ellipsoid elongation parameter and varies from 0.2 to 1.0. As elongated ellipsoids pack efficiently [9], the number of required ellipsoids to build the void structure decreases when m decreases.

The relative density tunes the void structure volume in the cellular material. It is calculated by summing all the voxel units belonging to the solid phase as follows

$$\rho = \sum i/V \tag{7}$$

where i is a solid phase unit (voxel) and V is the volume of cellular solid ($V = 100^3$). In this work, the relative density is varied from 0.1 to 0.3.

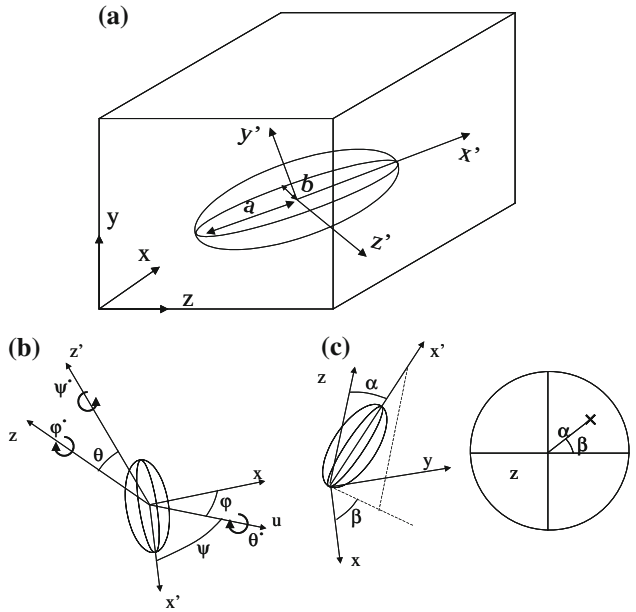


Fig. 1 (**a**) Illustration of ellipsoid orientation in 3D cellular solid with (**b**) Euler angles (ψ, θ, φ) describing the orientation of the ellipsoid void and (**c**) parameters (α, β) controlling the pole figure (signature of the ellipsoid orientation in 2D space). x, y and z refer to normal and transverse directions

The ellipsoid orientation is described using Euler angles (Fig. 1b). A given ellipsoid can be preferentially oriented in x direction or randomly spread in space under the angular limits

$$-\pi \leq (\psi, \varphi, \theta) \leq \pi \tag{8}$$

where ψ, φ and θ are Euler angles in radians.

Two populations of ellipsoid distributions are then defined based on whether they are fully or random oriented. The ratio of fully oriented ones ϕ is then varied from 0 to 1. In order to control the randomness of the ellipsoid distribution, the pole figure, representing the 2D trace of the ellipsoid orientations in space, is considered (Fig. 1c). The pole figure is characterised by the polar coordinates (α, β). Note that alignment of ellipsoids in the x direction corresponds to (α, β) = (π/2, 0). Figure 2 illustrates examples of typical generated structures. In Fig. 2a, ellipsoids are fully oriented in the x direction ($\phi = 1.0$). The same figure depicts the increase of the void elongation with the decrease of m. In Fig. 2b, the ratio of ellipsoids aligned in the x direction is varied for a fixed m value. The pole figures corresponding to the structures shown in Fig. 2b are also plotted where the point representing the aligned ellipsoids is arrowed in each case. The number shown near this point is ϕ * NB, i.e., the number of aligned ellipsoids. Note that the random character of orientations is still guaranteed for $\phi = 0.6$.

Structure characteristics

As the relative density of the material is low (between 0.1 and 0.3), the open cell character characterises the generated cellular solids. However, this characteristic introduces a difficulty to perform the FE calculations because elements must be at least connected face to face. It is not surprising to find discontinued parts in the solid (Fig. 3). These have to be removed to avoid rigid body displacements. It is interesting to study the correlation between the amount of discontinuity and the generation parameters (ρ, NB, ϕ, m).

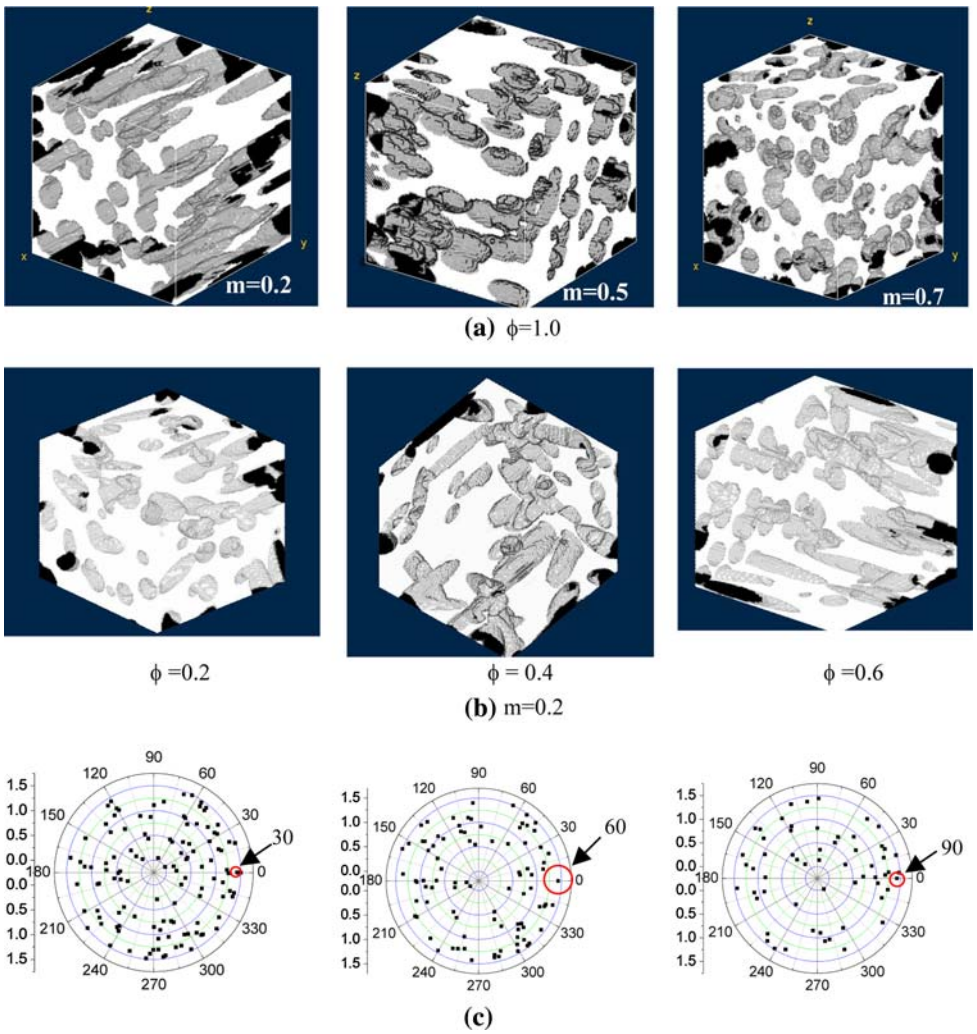
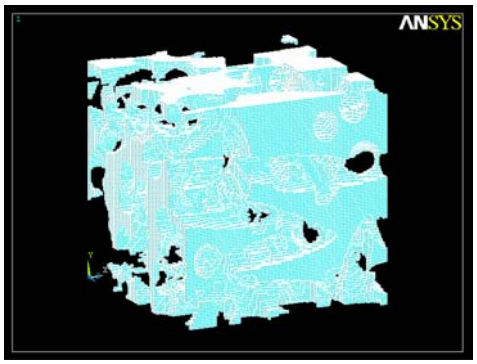
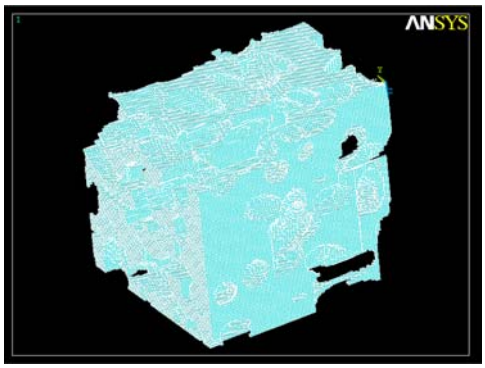


Fig. 2 Typical cellular structures built using ellipsoid shape voids with different ellipsoid distributions and the corresponding pole figures. (**a**) Ellipsoids are fully oriented in the x direction ($\phi = 1.0$) and elongation parameter (m) is varied. (**b**) Two populations of ellipsoids, one oriented in x direction (ϕ varied) and the other randomly spread in all space directions ($m = 0.2$, NB = 150). (**c**) Pole figures corresponding to the structures shown in Fig. 2b detailing the number of oriented ellipsoids in the x direction. Relative density for all structures is around 0.23



(a)



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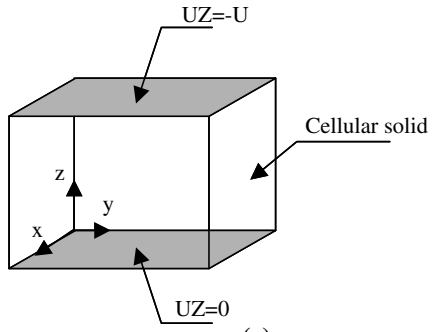

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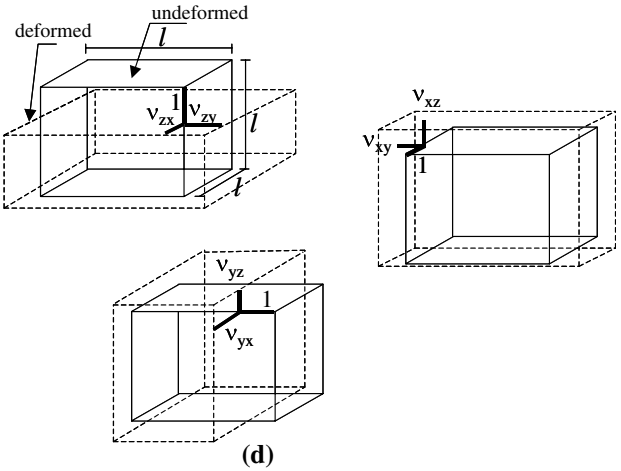

(d)

Fig. 3 Regular meshing of typical open cell structures (**a**) $\phi = 0.3$, $m = 0.2$ (2.1×10^5 cubic elements with discontinuity effect). (**b**) $\phi = 0.4$, $m = 0.42$ (3.1×10^5 cubic elements without discontinuity effect). (**c**) Typical boundary conditions for compression in the z direction. (**d**) Poisson ratios computation in the main loading conditions. For illustration purposes, the axial displacement is unity

Figure 4 shows the evolution of the amount of discontinuities as function of relative density calculated with respect to the largest connected feature in the cellular solid.

It is evident to find an increase of the discontinuity ratio with the decrease of the relative density irrespective of other generation parameters. The amount of discontinuity represents 0.35–0.4 of the solid phase for ρ lower than 0.1 and falls down to 0.15 for ρ larger than 0.30. When ellipsoids are all randomly spread (Fig. 4a), the discontinuity ratio is found larger for a given relative density compared to the case when ellipsoids are aligned (Fig. 4b). This means that the disorder in the material increases with the decrease of ϕ for a decreasing m value. It is also interesting to note that when fixing m, no influence of ϕ can be predicted (Fig. 4c). Thus, the decrease of discontinuities in the solid can be correlated to the product ϕm, where NB increases with the increase of m.

After removing the discontinued parts, it is important to check the stability of the anisotropy created using (ϕ, m). In particular, the wall thickness in the percolating solid has to be sensitive to space directions. Figure 5 shows cumulative distributions of the wall thickness as function of space directions for the largest feature in the cellular solid. In the case of fully oriented ellipsoids ($\phi = 1.0$), the cumulative distribution of the wall thickness in the x direction is larger than that in the transverse directions (Fig. 5a). This means that the percolation of the solid phase in the x direction is larger than that in the transverse directions. In the case of randomly oriented ellipsoids (Fig. 5b), all distribution trends are confused showing that the frequency of a given wall thickness is equivalently represented in all space directions.

Finite element method

The largest percolating object is meshed according to the scheme shown in Fig. 3b using isotropic cubic elements. These elements are defined by 8 nodes on a grid containing 100 voxels per edge, where each voxel is replaced by a volume element. A fixed mesh density is used for all relative densities and the number of elements is then $\rho * V$. The solid phase is given arbitrary but equal Young's moduli and the solid phase Poisson ratio is fixed to 0.35.

The elastic energy is minimised, through an iterative process, using a preconditioned conjugate gradient solver

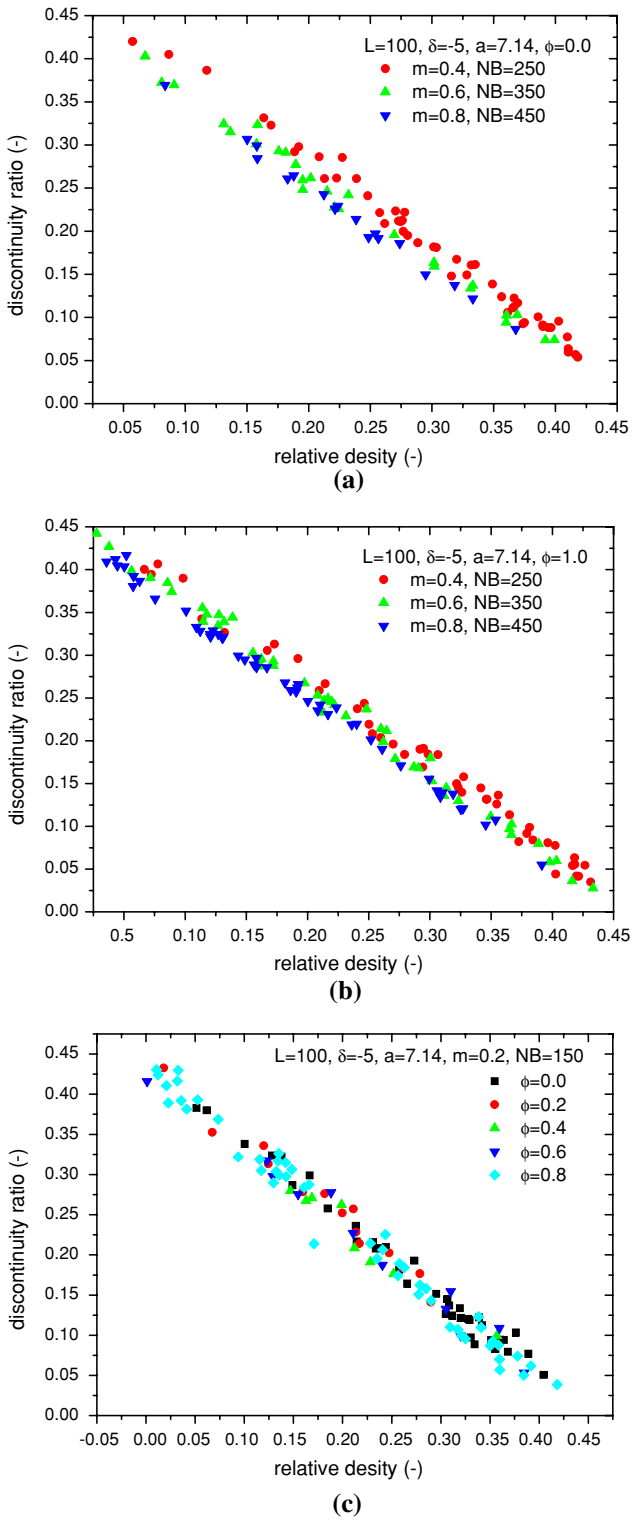

Fig. 4 Discontinuity effects in open cellular structures. (**a**) $\phi = 0.0$, m varied. (**b**) $\phi = 1.0$, m varied. (**c**) ϕ varied, $m = 0.2$

(PCG) under the ANSYS code. The loading conditions correspond to a constrained face against displacement in the loading direction (UZ = 0 for a loading in the z direction) and a displaced opposite face by a negative amount in the same loading direction (UZ = −U) to simulate a compression test (Fig. 3c). Thus, both faces are free of restrictions for the lateral displacements. To avoid rigid body displacement, some nodes of the lateral faces are constrained against the lateral displacement. The uniaxial test is simulated in the three normal directions X, Y and Z. The Poisson coefficients are evaluated by estimating the average displacement of the lateral faces with respect to the imposed displacement in the loading direction. For example, a loading in the x direction permits to evaluate 2 Poisson ratios v_{xy} and v_{xz} (Fig. 3d). For each couple (m, ϕ) where $m = \{0.2, 0.4, 0.6, 0.8, 1.0\}$ and $\phi = \{0.0, 0.2, 0.4, 0.6, 0.8, 1.0\}$, at least five structures are used with ρ ranging from 0.1 to 0.3. With 3 runs per condition, the total number of FE jobs is 450. The calculation time per condition depends on the relative density and approaches roughly 20 min on a 3.59 GHz PC with 3GHz of RAM.

Results and discussion

The predicted results show that all Poisson ratios are not independent quantities and thus it is possible to defined three categories

a. Poisson ratios issued from loading the x direction (v_{xy}, v_{xz});
b. Poisson ratios calculated from a loading in the transverse directions (v_{yx}, v_{zx});
c. Poisson ratios corresponding to transverse displacements and transverse loadings (v_{yz}, v_{zy}).

Figure 6 shows the evolution of the three categories as function of relative density. The largest Poisson ratios are those corresponding to categories a and c. The smallest ones are those corresponding to category b where displacements in the x direction are calculated from a loading in the transverse directions. A large scatter characterises the predicted results. Large values ($v > v_s$ where $v_s = 0.35$ is the solid phase Poisson ratio) are mainly explained by an effect of anisotropy cumulated to structural boundary effects. In some cases, excessive rotation of nodes belonging to the loaded face causes wrong Poisson coefficient values. This particularly happens with unrealistic cellular solid shapes after removing the disconnected parts. In the work of Silva et al. [10], nodes belonging to the loaded and unloaded sides were constrained against rotation to avoid surface crushing dominating mechanism. In the work of Chen et al. [11], the use of periodic boundary conditions was suggested to correct these effects and led to predictions much closer to the analytical results [11]. However, the use of periodic boundary conditions suppose that nodes belonging to the solid surfaces are

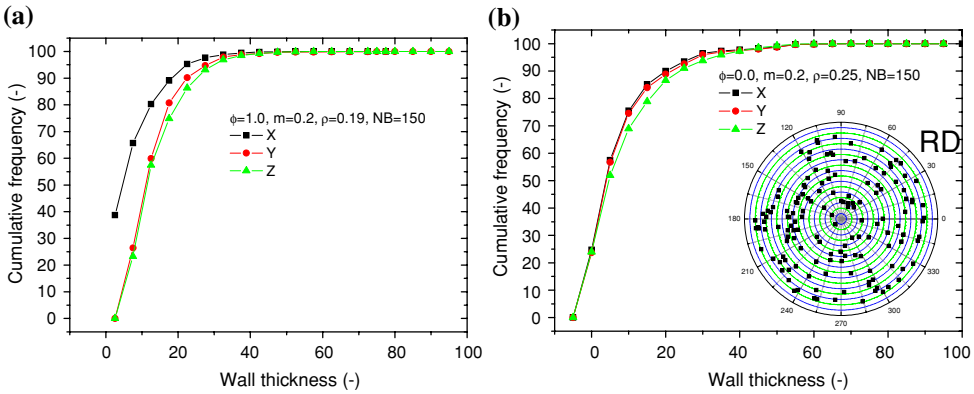

Fig. 5 Wall thickness data for typical generated cellular solids. Cumulative distribution of wall thickness as function of space direction (**a**) case of fully oriented voids. (**b**) case of fully randomly oriented voids

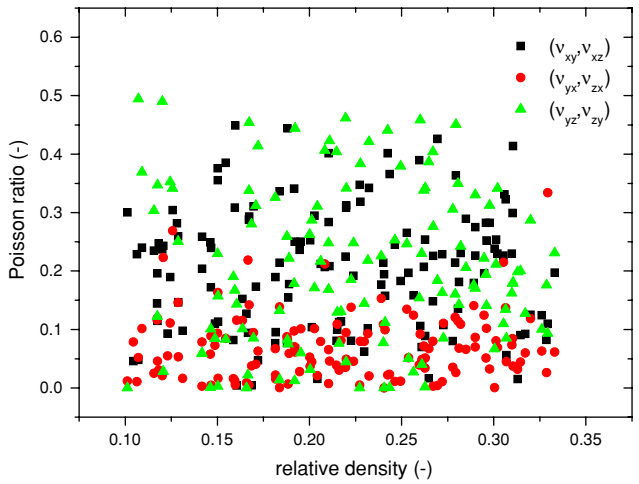

Fig. 6 Poisson ratio data as function of relative density (all conditions (ϕ, m) are confused)

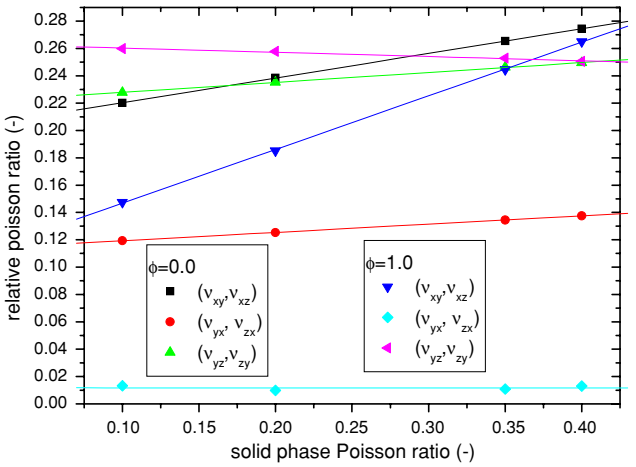

Fig. 7 Relative Poisson ratio as function of solid phase Poisson ratio for two different ellipsoid distributions ($\phi = 1.0$ and $\phi = 0.0$). The other parameters are ($m = 0.2$, $\rho = 0.31$, NB $= 150$)

exactly located at the same lateral positions. In our case, data from such cases are removed from the analysis.

It seems that no clear correlation exists between relative density and Poisson ratios and thus only the correlations between generation parameters (ϕ, m) and the Poisson ratio categories can be studied. The point of density independence was already addressed in the paper of Roberts et al. [6]. Authors attribute the constant effective Poisson ratio at low densities to the prediction of a fixed point irrespective of the cellular structure [4]. The physical sense behind such a fixed point is not yet addressed.

Despite the data scatter in Fig. 6, the prediction of two levels is undoubtedly related to the geometry of the cellular solid. In our case, the evolution of relative Poisson ratio as function of intrinsic Poisson ratio reveals an anisotropy in the parameter sets depending on ϕ parameter. The evolution of the Poisson coefficient values for two different ϕ values is depicted in Fig. 7. The data given here are averages of at least four generated patterns. The linear tendencies are added based on a linear fitting model with an excellent correlation factor ($R^2 > 0.99$) except for the case (($\phi = 1$, (ν_{yx}, ν_{zx})). The scatter of the data is however large

Table 1 Linear correlations between generation parameters and effective properties

Parameter	Variable		
	v_0	b_1	b_2
ν_{xy}, ν_{xz}	0.257	−0.028	0.061
ν_{yx}, ν_{zx}	0.066	0.043	−0.046
ν_{yz}, ν_{zy}	0.235	0.017	0.074

$y = v_0 + b_1 m + b_2 \phi$

(30% on the average irrespective of the generation set). It is found that the most sensitive effective parameter to ν_s is the set (ν_{xy}, ν_{xz}). For $\phi = 1.0$, the solid phase becomes fully connected in the direction of cell alignment. If we compute the relative density attached to a particular line (the number of cell walls per unit length), then the linear relative density is larger in the direction of cell alignment compared to the same quantity in the transverse direction.

Knowing that at large relative densities ν becomes dependent on ν_s means that the set (ν_{xy}, ν_{xz}) is suggested to

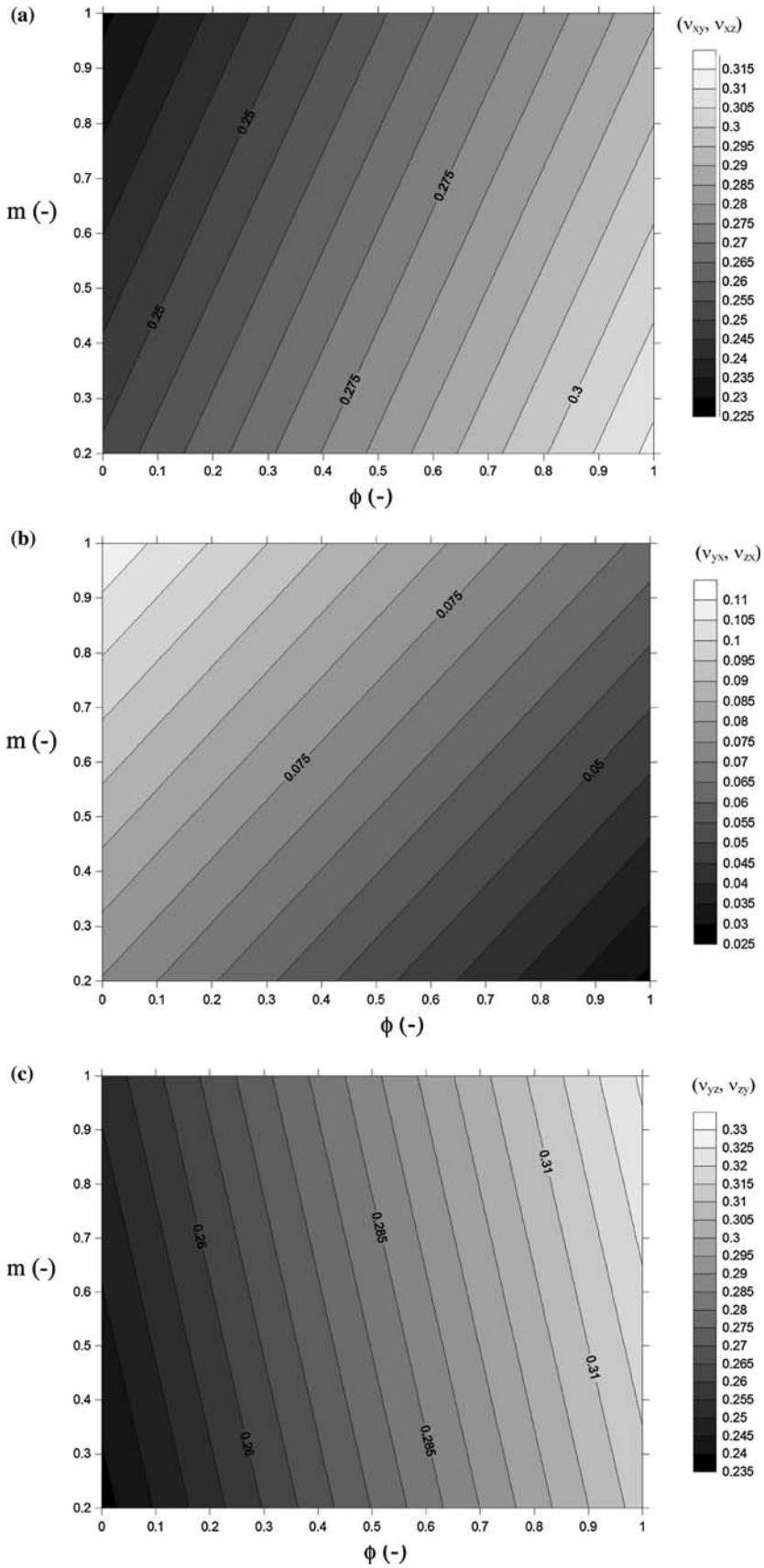

Fig. 8 Poisson ratio coefficients as function of generation parameters. (**a**) (v_{xy}, v_{xz}), (**b**) (v_{yx}, v_{zx}) and (**c**) (v_{yz}, v_{zy})

vary significantly when increasing ϕ. Obviously, the set (ν_{yx}, ν_{zx}) is ν_s independent because of the weak solid connectivity in the transverse directions. The set (ν_{yz}, ν_{zy}) exhibits a slight dependence on ϕ parameter because both loading and lateral displacements are in the transverse directions.

Automatic fitting routines are used to study the influence of ϕ and m on the Poisson ratio evolutions. The linear fit of the predicted data indicates that simple equations provide only an approximate description of the predicted data. This is mainly due to the scatter of the Poisson ratio results. One way to understand the origin of this scatter is to analyse the varied number of cellular solids obtained at a given relative density. Following the result shown in Fig. 4, the range of solid phase discontinuities suggests that a variety of different cellular structures can be obtained around a given relative density. Despite the approximate nature of the correlation found using linear interpolation, the result can be understood qualitatively. The derived equation can be expressed in the following form

$$\left(\nu_{ij}, \nu_{Kl}\right) = \nu_0 + b_i * m + b_2\phi \tag{9}$$

where the couple (ν_{ij}, ν_{kl}) relates to one of the Poisson ratio categories. ν_0, b_1 and b_2 are constant parameters determined using the fitting routine and shown in Table 1.

Figure 8 shows the predicted maps relating the average Poisson coefficients belonging to each category as function of ϕ and m parameters. All values are, on the average, less than the Poisson coefficient of the solid phase.

In Fig. 8a, the couple (ν_{xy}, ν_{xz}) is found to vary on the average between 0.23 and 0.32 and increases with the increase of the ratio ϕ/m. This correlation can be explained by the solid phase percolation in the x direction, which increases with the increase of m and the decrease of ϕ. In the counter part (ν_{yx}, ν_{zx}) is found to behave oppositely because of the material disorder in the transverse directions (Fig. 8b). The range of variation is limited to (0.03–0.11), where the upper bound gives an underestimation of the Poisson coefficient at isotropy conditions ($m = 1.0$).

In Fig. 8c, the range of variation of (ν_{yz}, ν_{zy}), namely (0.24–0.33), is equivalent to that of (ν_{xy}, ν_{xz}). However (ν_{yz}, ν_{zy}) is found to increase with the increase of ϕm with less sensitivity to m (Table 1). This result indicates that this category is sensitive to the amount of disorder created in the transverse directions and that in turn is controlled by the product ϕm (Fig. 4).

Conclusions

Poisson ratios of a cellular solid subject to a structural anisotropy can be categorised into three main sets corresponding to (ν_{xy}, ν_{xz}), (ν_{yx}, ν_{zx}) and (ν_{yz}, ν_{zy}). It is found that the sets are not sensitive to relative density in the low range approximation (ρ between 0.1 and 0.3). The amount of discontinuities of the solid phase in the cellular solid may explain this correlation because of the large scatter of the Poisson ratios obtained for a given relative density. The predicted values, for a given set of parameters, differ depending on the main characteristics of the largest percolating solid phase feature. It is predicted that the category (ν_{xy}, ν_{xz}) depends significantly on the value of Poisson ratio if the ratio of aligned ellipsoids (ϕ parameter) is large enough. The lowest Poisson ratio values are those corresponding to the set (ν_{yx}, ν_{zx}). The linear approximation gives a qualitative interpretation of the Poisson ratio evolutions especially for the solids with elongated ellipsoid voids (small m values) suggesting that the product ϕm or the ratio ϕ/m are reduced influential parameters. The transverse quantities (ν_{yz}, ν_{zy}) are found to be correlated to the product ϕm, which controls the amount of discontinuity in the solid and thus the disorder in the material.

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